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Discontinuous media and underdetermined scattering problems

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Abstract. The problem of ambiguities in trying to determine a shape by means of scattering experiments, with one or a few illuminating angles and all directions of receivers, is discussed by means of numerical experiments. The model equation gives a good representation of scattering of scalar waves which can take into account impedance discontinuities inside the scatterer. Physical problems include, for instance, acoustical waves in media where the density ρ and the Lamé parameter λ may vary continuously everywhere except across a finite number of smooth surfaces through which they jump. For the sake of simplicity, results are illustrated here in the two-dimensional case, with one discontinuity curve. The input is a closed curve of arbitrary shape, with arbitrary boundary conditions, chosen in such a way that the quadratic approximation (Born term+second-order term) is valid. The scattering amplitude is calculated for one incident angle. Then a circular curve is calculated, with appropriate boundary conditions, which yields the same scattering amplitude within the approximation. Variations of incident angles, frequencies and shapes are discussed for the calculated examples. The relevance of these results in the theory of non-destructive sensing is obvious.

1. Introduction

The problem of ambiguities in trying to determine the discontinuity shapes inside a material by means of scattering experiments with one or a few sources and all directions of receivers, is of obvious physical interest as well as importance in applications. In a previous paper [7], Sabatier gave the scattering theory corresponding to the impedance equation, with discontinuity surfaces corresponding to a jump in impedance and/or its normal derivative. Between these surfaces, the material was assumed to be inhomogeneous, but with a smooth variation of parameters only, such that the impedance was twice differentiable. This model is recalled in section 2 and adapted to the two-dimensional case, which was not treated in [1]. In particular we give the formulae for Born and quadratic approximations for the scattering amplitude when the jumps of impedances and derivatives across the discontinuity curves are small, together with their variations in continuous parts. For the sake of simplicity the numerical experiments, described in section 3, focus on the case of one discontinuity curve and no impedance variation elsewhere. Equivalent shapes (with appropriate discontinuities of impedance and derivatives) are shown and discussed.

2. The scattering problem

We start from the impedance equation

$$(\alpha^{-2}\operatorname{div}\alpha^2\operatorname{grad}+k^2-V(x))\varphi(k,x)=0 \tag{2.1}$$

where $k, x \in \mathbb{R}^2$, $\alpha > 0$ is defined inside domains $\Omega_0, \Omega_1, \ldots, \Omega_{N+1}$, such that $\Omega_i \cap \Omega_j = \varnothing$ for any $i \neq j$, $\mathbb{R}^2 = \Sigma_0^{N+1} \bar{\Omega}_i$, and $\delta\Omega_i = S_i$ is the external boundary of Ω_i, and the internal boundary of Ω_{i+1}. The domains are ordered from Ω_0 to Ω_{N+1} and all finite except Ω_{N+1}, which extends to infinity in all directions. In addition we assume that each S_i is $\mathscr{C}^2$, and α is $\mathscr{C}^2$ inside $\mathbb{R}^2 \backslash S$, where $S = \bigcup_{i=1}^N S_i$, with $\alpha(x)$ and $\partial\alpha(x)/\partial\nu_x$ going to finite limits at any point $x_i \in S_i$, and ν being a normal vector to S_i pointing outward, i.e. in Ω_{i+1} towards S_{i+1}. At any point $x_p \in S_p$ labelling the + and − sides of ν_p as external and internal parts, we can characterize the jump of α and its derivative throughout S_p by the following 'singular data':

transmission and reflection factors

$$\frac{1}{t_p} = \frac{1}{2}\left[\frac{\alpha_p^+}{\alpha_p^-} + \frac{\alpha_p^-}{\alpha_p^+}\right] \qquad \frac{r_p}{t_p} = \frac{1}{2}\left[\frac{\alpha_p^+}{\alpha_p^-} - \frac{\alpha_p^-}{\alpha_p^+}\right]$$

slope factor

$$\frac{\bar{s}_p}{t_p} = \frac{1}{2}\,\nu \cdot \left[\frac{\operatorname{grad} \alpha_p^-}{\alpha_p^+} - \frac{\operatorname{grad} \alpha_p^+}{\alpha_p^-}\right].$$

It is easy to see that if α and β have the same singular data at x_p on S_p, α/β is continuous across S_p at x_p together with $(\alpha \operatorname{grad} \beta - \beta \operatorname{grad} \alpha) \cdot \nu$. The solution ϕ is to be continuous through $\mathbb{R}^2$, together with $\alpha^2(\partial\phi/\partial\nu)$. Now the basic equivalences in this problem are expressed by the following theorems, where α and ϕ defined as above:

Theorem 1. If $\beta > 0$ is $\mathscr{C}^2(\mathbb{R}^2 \backslash S)$, where $S = \bigcup_{i=1}^N S_i$, whereas α/β is continuous through S together with $\alpha(\partial\beta/\partial\nu) - \beta(\partial\alpha/\partial\nu)$, and $\alpha\Delta\beta - \beta\Delta\alpha = 0$ at any $x \in \mathbb{R}^2 \backslash S$, then $\psi := \alpha\varphi/\beta$ is a solution of

$$(\beta^{-2} \operatorname{div} \beta^2 \operatorname{grad} + k^2 - V(x))\psi(k, x) = 0 \qquad x \in \mathbb{R}^2 \backslash S \tag{2.2}$$

α and β are called 'standard equivalent', i.e. they correspond to the same scattering problem.

Theorem 2. The function $\psi := \alpha\varphi$ is a solution of the chain of Schrödinger equations

$$(\Delta + k^2 - V - \alpha^{-1}\Delta\alpha)\psi(k, x) = 0 \qquad x \in \mathbb{R}^2 \backslash S \tag{2.3}$$

linked by the

$$\text{continuity through } S \text{ of } \psi/\alpha \text{ and } \alpha(\partial\psi/\partial\nu) - \psi(\partial\alpha/\partial\nu). \tag{2.4}$$

The impedance scattering problem was studied by Sabatier [1] in the three-dimensional case. Here we write the results in the two-dimensional case, because the result we want to show belongs to this case. Because of theorem 2, there always exist two equivalent formulations of the same physical problem and it is useful to go back and forth from one to the other. In the impedance formulation of the scattering problem, $\varphi(k, x)$ is sought such that

$$(\alpha^{-2} \operatorname{div} \alpha^2 \operatorname{grad} + k^2 - V(x))\varphi(k, x) = 0 \qquad x \in \mathbb{R}^2 \tag{2.5}$$

$\varphi_s := \alpha(x)\varphi(k, x) - \exp[\mathrm{i}k \cdot x]$ is Sommerfeld outgoing, i.e. in two-dimensional cases

$$\left(\frac{x}{|x|} \cdot \operatorname{grad} \varphi_s(x)\right) - \mathrm{i}k\varphi_s(x) = \mathrm{o}\left(\frac{1}{|x|^{1/2}}\right) \qquad |x| \to \infty. \tag{2.6}$$

In the Schrödinger chain formulation, $\varphi(k, x)$ is sought such that

$$(\Delta+k^2-V-\alpha^{-1}\Delta\alpha)\psi(k,x)=0 \qquad x\in\mathbb{R}^2\backslash S \tag{2.7}$$

$$\left.\begin{aligned}&\alpha^+(x)\frac{\partial\psi^+(x)}{\partial\nu}-\psi^+(x)\frac{\partial\alpha^+(x)}{\partial\nu}=\alpha^-(x)\frac{\partial\psi^-(x)}{\partial\nu}-\psi^-(x)\frac{\partial\alpha^-(x)}{\partial\nu}\\&\psi^+(x)/\alpha^+(x)=\psi^-(x)/\alpha^-(x) \qquad x\in S\end{aligned}\right. \tag{2.8}$$

$$\psi(k,x)-\exp[\mathrm{i}k\cdot x] \text{ is Sommerfeld outgoing.} \tag{2.9}$$

As in the three-dimensional case, it is possible to prove that one can construct a Green function that corresponds to discontinuities without potentials, i.e. which is a solution of

$$(\Delta_x+k^2)G(x,y)=-\delta(x-y) \qquad x,y\in\mathbb{R}^2 \tag{2.10}$$

completed by (2.8) and the Sommerfeld condition (2.6). This Green function can be constructed by means of the Helmholtz Green function

$$\Phi(x,y)=\frac{\mathrm{i}}{4}H_0^{(1)}(|k||x-y|) \qquad x,y\in\mathbb{R}^2, x\neq y \tag{2.11}$$

and by solving surface integral equations with compact operators, i.e. $G(x, y)$ is identified as the resolvent kernel of the following system.

$$(\Delta_x+k^2)u(x)=-f(x) \qquad x\in\mathbb{R}^2\backslash S \tag{2.12}$$

$$\alpha^+(x)\frac{\partial u^+(x)}{\partial\nu}-u^+(x)\frac{\partial\alpha^+(x)}{\partial\nu}=\alpha^-(x)\frac{\partial u^-(x)}{\partial\nu}-u^-(x)\frac{\partial\alpha^-(x)}{\partial\nu} \tag{2.13}$$

$$u^+(x)/\alpha^+(x)=u^-(x)/\alpha^-(x) \qquad x\in S$$

$$u(x) \text{ is Sommerfeld outgoing (2.6).} \tag{2.14}$$

$u(x)$ reads

$$u(x)=F(x)+\sum_{j=1}^{N}\int_{S_j}\mathrm{d}s(z)\Phi(z,x)\phi_j(z)+\sum_{j=1}^{N}\int_{S_j}\mathrm{d}s(z)\frac{\partial\Phi}{\partial\nu_z}(z,x)\psi_j(z) \tag{2.15}$$

with

$$F(x)=\int_{\mathbb{R}^2}\mathrm{d}s(z)\Phi(z,x)f(z) \tag{2.16}$$

$$u(x)=\int_{\mathbb{R}^2}\mathrm{d}s(z)G(z,x)f(z). \tag{2.17}$$

As in the three-dimensional case $u(x)$ satisfies (2.12) and the Sommerfeld condition. We determine ϕ_j and ψ_j with the continuity conditions (2.13) and we have

$$\begin{cases}\psi=2\beta F+\beta\boldsymbol{S}\phi+\beta\boldsymbol{K}\psi\\ \phi+\gamma\psi=2\beta F'-2\beta' F+\beta\boldsymbol{T}\psi-\beta'\boldsymbol{K}\psi+\beta\boldsymbol{K}'\phi-\beta'\boldsymbol{S}\phi\end{cases} \tag{2.18}$$

with

$$\psi_j=\psi \qquad \text{for } x\in S_j$$

$$\beta(x)=\frac{\alpha^+(x)-\alpha^-(x)}{\alpha^+(x)+\alpha^-(x)} \qquad \beta'(x)=\frac{\alpha'^+(x)-\alpha'^-(x)}{\alpha^+(x)+\alpha^-(x)}$$

$$\gamma(x)=\frac{\alpha'^+(x)+\alpha'^-(x)}{\alpha^+(x)+\alpha^-(x)}$$

and the surface operators $\boldsymbol{S}, \boldsymbol{K}, \boldsymbol{K}', \boldsymbol{T}$ defined from a surface S_i to S_j by:

$$(S_{ij}f)(x)=2\int_{S_j} \mathrm{d}s(z)\Phi(z,x)f(z) \qquad x\in S_i \tag{2.19}$$

$$(K_{ij}f)(x)=2\int_{S_j} \mathrm{d}s(z)\frac{\partial\Phi}{\partial\nu_z}(z,x)f(z) \qquad x\in S_i \tag{2.20}$$

$$(K'_{ij}f)(x)=2\int_{S_j} \mathrm{d}s(z)\frac{\partial\Phi}{\partial\nu_x}(z,x)f(z) \qquad x\in S_i \tag{2.21}$$

$$(T_{ij}f)(x)=2\frac{\partial}{\partial\nu_x}\int_{S_j} \mathrm{d}s(z)\frac{\partial\Phi}{\partial\nu_z}(z,x)f(z) \qquad x\in S_i. \tag{2.22}$$

We have the following result:

Theorem 3. We assume that $\beta, \beta', \gamma, \psi$ are in $\mathscr{C}^1(S)$ and ϕ in $\mathscr{C}(S)$. If $N(1-\beta\boldsymbol{K})=0$ and $N(1-\boldsymbol{B}/(1+\beta^2))=0$, then the system (2.18) has a unique solution

$$\psi=(1-\beta\boldsymbol{K})^{-1}\beta(2F+\boldsymbol{S}\phi) \tag{2.23}$$

$$\phi=[(1+\beta^2)\mathbf{1}-\boldsymbol{B}]^{-1}\boldsymbol{A}F \tag{2.24}$$

with:

$$\boldsymbol{A}F=2\beta F'-2[\beta'+(\gamma\mathbf{1}+\beta'\boldsymbol{K}-\beta\boldsymbol{T})(1-\beta\boldsymbol{K})^{-1}\beta]F \tag{2.25}$$

$$\boldsymbol{B}=-(\gamma\mathbf{1}+\beta'\boldsymbol{K})(\mathbf{1}-\beta\boldsymbol{K})^{-1}\beta\boldsymbol{S}+\beta\boldsymbol{K}'-\beta'\boldsymbol{S}+\boldsymbol{C} \tag{2.26}$$

where the operator $\boldsymbol{C}$ is complex and $\mathrm{O}(\|\beta\|^2)$ as $\|\beta\|\to 0$.

That $\boldsymbol{A}$ and $\boldsymbol{B}$ are compact operators was proved in the three-dimensional case [1] and can be proved in the same way for two dimensions. Then one can use $G(x,y)$ in order to derive the Lipmann–Schwinger equation of the problem

$$\psi(k,x)=\psi_{\mathrm{in}}(k,x)-\int_{\mathbb{R}^2\setminus S} G(x,y)[V(y)+\alpha^{-1}\Delta\alpha(y)]\psi(k,y)\,\mathrm{d}y \tag{2.27}$$

where $x\in\mathbb{R}^2\setminus S$ and

$$\psi_{\mathrm{in}}(k,x)=\int_{S_N^+}\left[G(x,y)\frac{\partial\exp[\mathrm{i}k\cdot y]}{\partial\nu_y}-\exp[\mathrm{i}k\cdot y]\frac{\partial G(x,y)}{\partial\nu_y}\right]\mathrm{d}s(y) + \begin{cases}\mathrm{e}^{\mathrm{i}k\cdot y} & \text{if } x\in\Omega_{N+1}\\ 0 & \text{otherwise.}\end{cases} \tag{2.28}$$

Hence the scattering is separated into two steps, the scattering by discontinuities, and that due to diffuse scattering in the presence of discontinuities. The former is described by ψ_{in}, which is equal to (2.28) but can also be obtained as the solution of

$$\begin{aligned}&(\Delta+k^2)\psi_{\mathrm{in}}(k,x)=0 \qquad x\in\mathbb{R}^2\setminus S\\ &\psi_{\mathrm{in}}/\alpha \text{ and } \alpha\frac{\partial\psi_{\mathrm{in}}}{\partial\nu}-\psi_{\mathrm{in}}\frac{\partial\alpha}{\partial\nu} \text{ continuous } / S\\ &\psi_{\mathrm{in}}(k,x)-\exp[\mathrm{i}k\cdot x] \text{ is Sommerfeld outgoing.}\end{aligned} \tag{2.29}$$

We have too, a relation between ψ_{in} and $G(x,y)$:

$$G(x,y)=\frac{\mathrm{e}^{\mathrm{i}[|k||x|+\pi/4]}}{(8\pi|k|)^{1/2}|x|^{1/2}}\psi_{\mathrm{in}}(-k\hat{x},y)+\mathrm{o}\left(\frac{1}{|x|^{1/2}}\right) \quad \text{'when } |x|\to\infty\text{'}. \tag{2.30}$$

Applying Green's theorem to $\psi_{\mathrm{in}}(k,y)$ and $\Phi(x,y)$ inside $|y|\leqslant R$, and letting $R\to\infty$, we obtain

$$\psi_{\mathrm{in}}(k,x)=\mathrm{e}^{\mathrm{i}k\cdot x}-\frac{1}{(8\pi|k|)^{1/2}}\frac{\mathrm{e}^{\mathrm{i}[|k||x|+\pi/4]}}{|x|^{1/2}}A_0(|k|\hat{x},k)+o\left(\frac{1}{|x|^{1/2}}\right). \tag{2.31}$$

Substituting this result in (2.27) yields

$$\psi(k,x)=\psi_{\mathrm{in}}(k,x)-\frac{\mathrm{e}^{\mathrm{i}[|k||x|+\pi/4]}}{(8\pi|k|)^{1/2}|x|^{1/2}}A_1(|k|\hat{x},k)+o\left(\frac{1}{|x|^{1/2}}\right)\tag{2.32}$$

and combining with (2.31) finally yields

$$\psi(k,x)=\mathrm{e}^{\mathrm{i}k\cdot x}-\frac{\mathrm{e}^{\mathrm{i}[|k||x|+\pi/4]}}{(8\pi|k|)^{1/2}|x|^{1/2}}A(|k|\hat{x},k)+o\left(\frac{1}{|x|^{1/2}}\right)\tag{2.33}$$

where $A=A_0+A_1$ is the sum of the scattering amplitude due to reflectors only and the diffuse scattering, the reflectors being present.

It is possible to trace back in the calculations the first and second-order terms with respect to the potentials size and to r_p, s_p, t_p. The procedure is the following: with the hypothesis of theorem 3, we obtain the expansion up to second order of ψ and ϕ

$$\begin{aligned}\phi=2\beta F'-2\beta' F+2\beta T\beta F-2\beta' K\beta F+2\beta K'\beta F'\\ -2\beta K'\beta' F-2\beta' S\beta F'+2\beta' S\beta' F+O(\|\beta\|^3)\end{aligned}\tag{2.34}$$

$$\psi=2\beta F+2\beta K\beta F+2\beta S\beta F'-2\beta S\beta' F+O(\|\beta\|^3)$$

where $\|\beta\|$ is $\mathrm{Sup}(\|\beta\|_{\mathscr{C}^1(S)},\|\beta'\|_{\mathscr{C}^1(S)})$ and it is easy to see that if $\|\beta\|$ is small enough, the conditions of validity of theorem 3 hold. We build $G(x,y)$ and deduce ψ_{in} with (2.30) and we get the scattering amplitude at the first (Born approximation) and second order (quadratic approximation).

Then one can reduce the result to its simplest form by using the standard equivalence to obtain the Born approximation

$$\begin{aligned}A(k',k)=-2\sum_{j=0}^{N}\int_{S_j}\mathrm{d}s(z)\beta(z)\frac{\partial}{\partial\nu_z}[\mathrm{e}^{\mathrm{i}(k-k')\cdot z}]+2\sum_{j=0}^{N}\int_{S_j}\mathrm{d}s(z)\beta'(z)\,\mathrm{e}^{\mathrm{i}(k-k')\cdot z}\\ +\int_{\mathbb{R}^2\setminus S}[V(y)+\alpha^{-1}\Delta\alpha(y)]\,\mathrm{e}^{\mathrm{i}(k-k')\cdot y}\,\mathrm{d}y\end{aligned}\tag{2.35}$$

and the quadratic approximation (only A_0 is given):

$$\begin{aligned}A_0(k',k)=&-2\int_S\mathrm{d}s(z)\frac{\partial}{\partial\nu_z}\mathrm{e}^{\mathrm{i}k\cdot z}\beta(z)\,\mathrm{e}^{-\mathrm{i}k\cdot z}\\
&+2\int_S\mathrm{d}s(z)\,\mathrm{e}^{\mathrm{i}k\cdot z}\left(\beta'(z)\,\mathrm{e}^{-\mathrm{i}k'\cdot z}-\beta(z)\frac{\partial}{\partial\nu_z}\mathrm{e}^{-\mathrm{i}k'\cdot z}\right)\\
&+4\int_S\mathrm{d}s(z)\,\mathrm{e}^{\mathrm{i}k\cdot z}\beta(z)\int_S\mathrm{d}s(t)\frac{\partial}{\partial\nu_z}\Phi(z,t)\left(\beta'(t)\,\mathrm{e}^{-\mathrm{i}k'\cdot t}-\beta(t)\frac{\partial}{\partial\nu_t}\mathrm{e}^{-\mathrm{i}k'\cdot t}\right)\\
&-4\int_S\mathrm{d}s(z)\,\mathrm{e}^{\mathrm{i}k\cdot z}\beta'(z)\int_S\mathrm{d}s(t)\Phi(z,t)\left(\beta'(t)\,\mathrm{e}^{-\mathrm{i}k'\cdot t}-\beta(t)\frac{\partial}{\partial\nu_t}\mathrm{e}^{-\mathrm{i}k'\cdot t}\right)\\
&+4\int_S\mathrm{d}s(z)\frac{\partial}{\partial\nu_z}\mathrm{e}^{\mathrm{i}k\cdot z}\beta(z)\int_S\mathrm{d}s(t)\Phi(z,t)\left(\beta'(t)\,\mathrm{e}^{-\mathrm{i}k'\cdot t}-\beta(t)\frac{\partial}{\partial\nu_t}\mathrm{e}^{-\mathrm{i}k'\cdot t}\right)\\
&+4\int_S\mathrm{d}s(z)\,\mathrm{e}^{\mathrm{i}k\cdot z}\beta'(z)\int_S\mathrm{d}s(t)\frac{\partial}{\partial\nu_t}\Phi(z,t)\beta(t)\,\mathrm{e}^{-\mathrm{i}k'\cdot t}\\
&-4\int_S\mathrm{d}s(z)\frac{\partial}{\partial\nu_z}\mathrm{e}^{\mathrm{i}k\cdot z}\beta(z)\int_S\mathrm{d}s(t)\frac{\partial}{\partial\nu_t}\Phi(z,t)\beta(t)\,\mathrm{e}^{-\mathrm{i}k'\cdot t}\\
&-4\int_S\mathrm{d}s(z)\,\mathrm{e}^{\mathrm{i}k\cdot z}\beta(z)\frac{\partial}{\partial\nu_z}\int_S\mathrm{d}s(t)\frac{\partial}{\partial\nu_t}\Phi(z,t)\beta(t)\,\mathrm{e}^{-\mathrm{i}k'\cdot t}\end{aligned}\tag{2.36}$$

where β, β' and γ are, now and in the following

$$\beta(z)=\frac{1-t(z)+r(z)}{1+t(z)+r(z)} \qquad \beta'(z)=-\frac{s(z)}{t(z)} \qquad \gamma(z)=-\beta(z)\beta'(z) \tag{2.37}$$

3. Numerical experiments on ambiguities

As a first contribution to a study of ambiguities beyond the basic equivalences, we have undertaken a set of numerical experiments where only one discontinuity curve was assumed, no diffuse scattering, and the conditions of Born approximation hold (they are checked by an evaluation of the next order). Hence

$$A_0(k',k)=-2\int_S \mathrm{d}s(z)\beta(z)\frac{\partial}{\partial\nu_z}[\mathrm{e}^{\mathrm{i}(k-k')\cdot z}]+2\int_S \mathrm{d}s(z)\beta'(z)\,\mathrm{e}^{\mathrm{i}(k-k')\cdot z} \tag{3.1}$$

k is fixed, $|k|=|k'|$.

If β and β' were complex numbers, it would be easy to show that a proper continuation of β and β', obtained by solving Dirichlet and Neuman problems, could yield values of β and β' on a different contour $\tilde{S}$ such that the same A is obtained, provided that there is not a mode of the domain between S and $\tilde{S}$. But imposing β, β' to be real is difficult in this analytic approach and it is less complicated to proceed in a different way: our method of computation provides 'ambiguities' which are checked by solving the direct problem independently for the two supposedly equivalent cases. Hence, arguing lengthily on algorithm convergence is useless.

We assume that S and $\tilde{S}$ are star-shaped with respect to the same centre. Their equations are $R=R(\theta)$ and $R=\tilde{R}(\theta)$ where R, $\tilde{R}$ are periodic (2π). Hence

$$\begin{aligned}A(k,k')=2\mathrm{i}|k|&\int_0^{2\pi}\mathrm{d}\theta g_1(\theta)\cos(\theta'-\theta)\,\mathrm{e}^{-\mathrm{i}|k|R(\theta)\cos(\theta'-\theta)}\\&-2\mathrm{i}|k|\int_0^{2\pi}\mathrm{d}\theta g_2(\theta)\sin(\theta'-\theta)\,\mathrm{e}^{-\mathrm{i}|k|R(\theta)\cos(\theta'-\theta)}\\&-2\mathrm{i}|k|\int_0^{2\pi}\mathrm{d}\theta[h_1(\theta)+h_2(\theta)]\,\mathrm{e}^{-\mathrm{i}|k|R(\theta)\cos(\theta'-\theta)}\\&+2\int_0^{2\pi}\mathrm{d}\theta l(\theta)\,\mathrm{e}^{-\mathrm{i}|k|R(\theta)\cos(\theta'-\theta)}\end{aligned} \tag{3.2}$$

where

$$g_1(\theta)=R(\theta)\beta(\theta)\,\mathrm{e}^{\mathrm{i}|k|R(\theta)\cos\theta} \qquad g_2(\theta)=R'(\theta)\beta(\theta)\,\mathrm{e}^{\mathrm{i}|k|R(\theta)\cos\theta}$$
$$h_1(\theta)=g_1(\theta)\cos\theta \qquad h_2(\theta)=g_2(\theta)\sin\theta$$
$$l(\theta)=\sqrt{R(\theta)^2+R'(\theta)^2}\beta'(\theta)\,\mathrm{e}^{\mathrm{i}|k|R(\theta)\cos\theta}$$

and the same formula holds for $\tilde{R}, \tilde{\beta}, \tilde{\beta}'$. We note P the problem with the data $\{R(\theta), \beta(\theta)$ and $\beta'(\theta)\}$ and $\tilde{P}$ with $\{\tilde{R}(\theta), \tilde{\beta}(\theta)$ and $\tilde{\beta}'(\theta)\}$. It turns out that calculations are most easy if, starting from an arbitrary star-shaped curve, we arrive at a circle (in the figures 1 to 5, the initial problem P corresponds to the initial star-shaped curve, and the equivalent problem $\tilde{P}$ corresponds to a circle. They are respectively labelled by a and b in the figure number).

Thus, we have to determine $\tilde{\beta}(\theta)$ and $\tilde{\beta}'(\theta)$ and $\tilde{R}$ from the problem P data in order to obtain the same results for a given value of $|k|$ and all angles θ'. Calculating the Fourier series with respect to θ', we obtain for P

$$\begin{aligned}A_n=2(-\mathrm{i})^n\bigg\{&\frac{|k|}{2}\int_0^{2\pi}\mathrm{d}\theta g_1(\theta)(J_{n+1}(|k|R(\theta))-J_{n-1}(|k|R(\theta)))\,\mathrm{e}^{-\mathrm{i}n\theta}\\&-\mathrm{i}\frac{|k|}{2}\int_0^{2\pi}\mathrm{d}\theta g_2(\theta)(J_{n+1}(|k|R(\theta))+J_{n-1}(|k|R(\theta)))\,\mathrm{e}^{-\mathrm{i}n\theta}\\&-\mathrm{i}\frac{|k|}{2}\int_0^{2\pi}\mathrm{d}\theta g_1(\theta)J_n(|k|R(\theta))(\mathrm{e}^{-\mathrm{i}(n-1)\theta}+\mathrm{e}^{-\mathrm{i}(n+1)\theta})\\&-\frac{|k|}{2}\int_0^{2\pi}\mathrm{d}\theta g_2(\theta)J_n(|k|R(\theta))(\mathrm{e}^{-\mathrm{i}(n-1)\theta}-\mathrm{e}^{-\mathrm{i}(n+1)\theta}\\&+\int_0^{2\pi}\mathrm{d}\theta l(\theta)J_n(|k|R(\theta))\,\mathrm{e}^{-\mathrm{i}n\theta}\bigg\}\end{aligned} \tag{3.3}$$

and for $\tilde{P}$

$$\begin{aligned}\tilde{A}_n=2\pi\tilde{R}\bigg\{&2|k|J_n(J_{n+1}-J_{n-1})\tilde{\beta}_0+2J_n^2\tilde{\beta}'_0\\&+|k|\sum_{m=1}^{\infty}\mathrm{i}^m([(J_{n+1}-J_{n-1})J_{m+n}+J_n(J_{m+n+1}-J_{m+n-1})]\tilde{\beta}^*_m\\&+(-1)^n[(J_{n+1}-J_{n-1})J_{m-n}+J_n(J_{m-n+1}-J_{m-n-1})]\tilde{\beta}_m)\\&+2\sum_{m=1}^{\infty}\mathrm{i}^mJ_n(J_{m+n}\tilde{\beta}'_m+(-1)^nJ_{m-n}\tilde{\beta}'_m)\bigg\}\end{aligned} \tag{3.4}$$

with $J_n=J_n(|k|\tilde{R})$, the Bessel functions, and $\tilde{\beta}_m$, $\tilde{\beta}'_m$ the Fourier coefficients of $\tilde{\beta}(\theta)$ and $\tilde{\beta}'(\theta)$.

To simplify the problem and get real values we note:

$$B_n=A_n+A^*_{-n}\qquad C_n=A_n-A^*_{-n} \tag{3.5}$$

and we define $\tilde{B}_n$ and $\tilde{C}_n$ the same way.

The expressions for $\tilde{B}_n$ and $\tilde{C}_n$ are:

$$\begin{aligned}\tilde{B}_n=2\pi\tilde{R}\bigg\{&4|k|J_n(J_{n+1}-J_{n-1})\tilde{\beta}_0+4J_n^2\tilde{\beta}'_0+2|k|\sum_{m=1}^{\infty}(-1)^m\\&\times([(J_{n+1}-J_{n-1})J_{2m+n}+J_n(J_{2m+n+1}-J_{2m+n-1})]\tilde{\beta}^*_{2m}\\&+(-1)^n[(J_{n+1}-J_{n-1})J_{2m-n}+J_n(J_{2m-n+1}-J_{2m-n-1})]\tilde{\beta}_{2m})\\&+4\sum_{m=1}^{\infty}(-1)^mJ_n(J_{2m+n}\tilde{\beta}'^*_{2m}+(-1)^nJ_{2m-n}\tilde{\beta}'_{2m})\bigg\}\end{aligned} \tag{3.6}$$

$$\begin{aligned}\tilde{C}_n=2\pi\mathrm{i}\tilde{R}\bigg\{&4\sum_{m=0}^{\infty}(-1)^mJ_n(J_{2m+1+n}\tilde{\beta}'_{2m+1}+(-1)^nJ_{2m+1-n}\tilde{\beta}'_{2m+1})+2|k|\sum_{m=0}^{\infty}(-1)^m\\&\times([J_{n+1}-J_{n-1})J_{2m+1+n}+J_n(J_{2m+n+2}-J_{2m+n})]\tilde{\beta}^*_{2m+1}+(-1)^n\\&\times[(J_{n+1}-J_{n-1})J_{2m+1-n}+J_n(J_{2m-n+2}-J_{2m-n})]\tilde{\beta}_{2m+1})\bigg\}.\end{aligned} \tag{3.7}$$

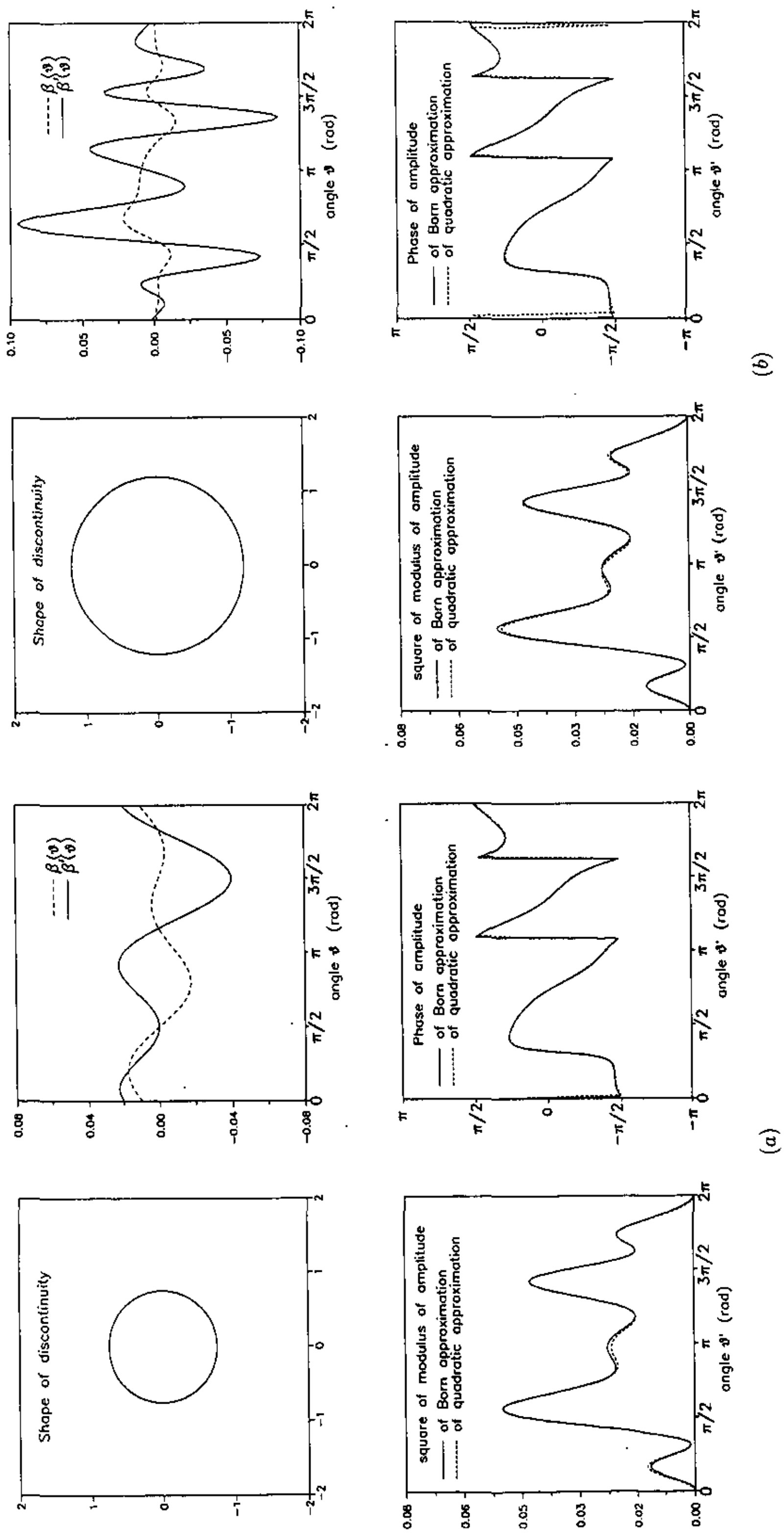


Figure 1. Scattering problems, $|k|=4$. Special data (a): $R=0.75$; $\beta=0.01(\sin 2\theta+\cos\theta)$; $\beta'=0.02(\cos 2\theta+\sin\theta)$. Radius in (b): $\bar{R}=1.2$.

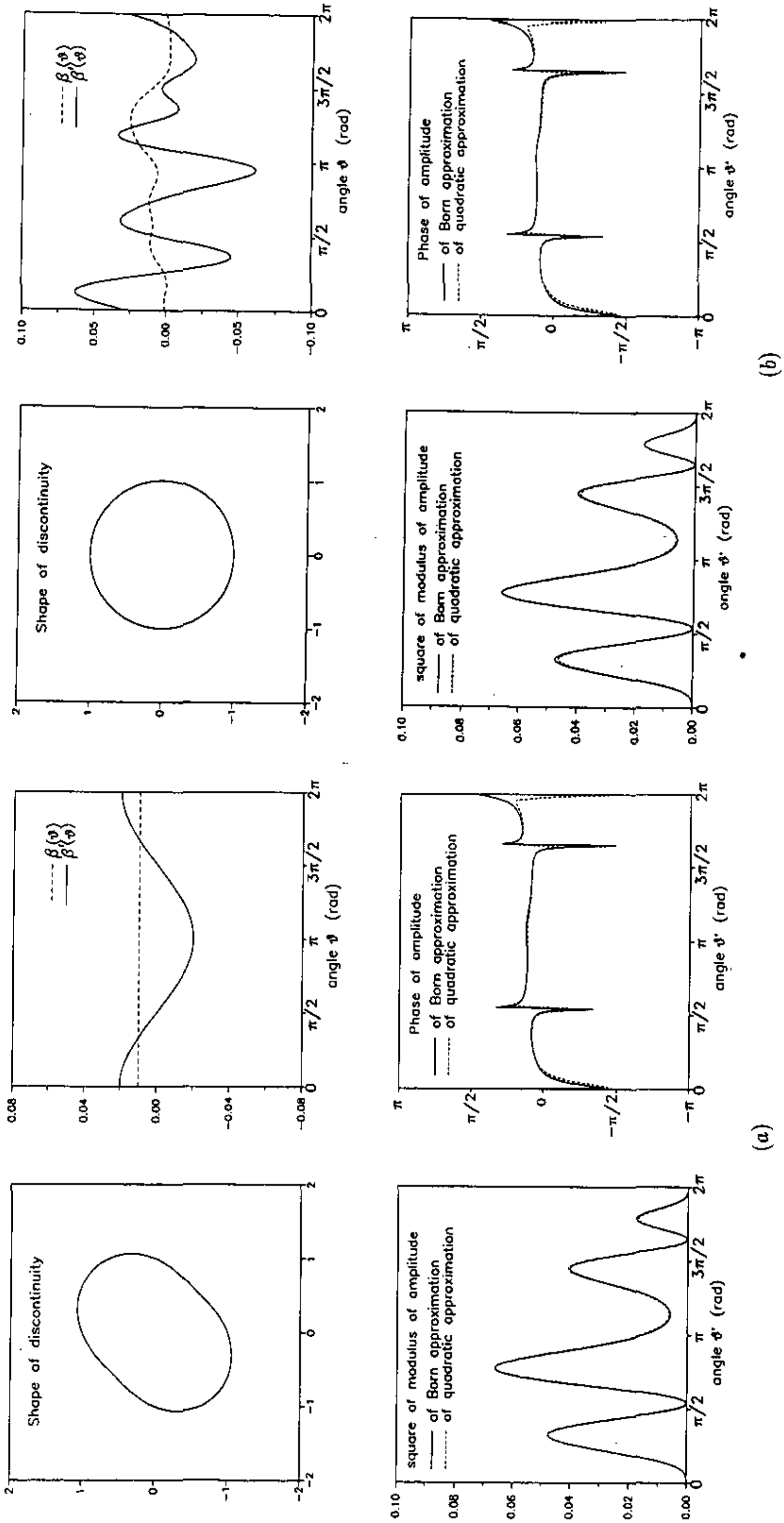


Figure 2. Scattering problems, $|k| = 3$. Special data (a): $R = 1 + 0.2 \sin 2\theta$; $\beta = 0.01$; $\beta' = 0.02 + \cos\theta$. Radius in ($b$): $\tilde{R} = 1.2$.

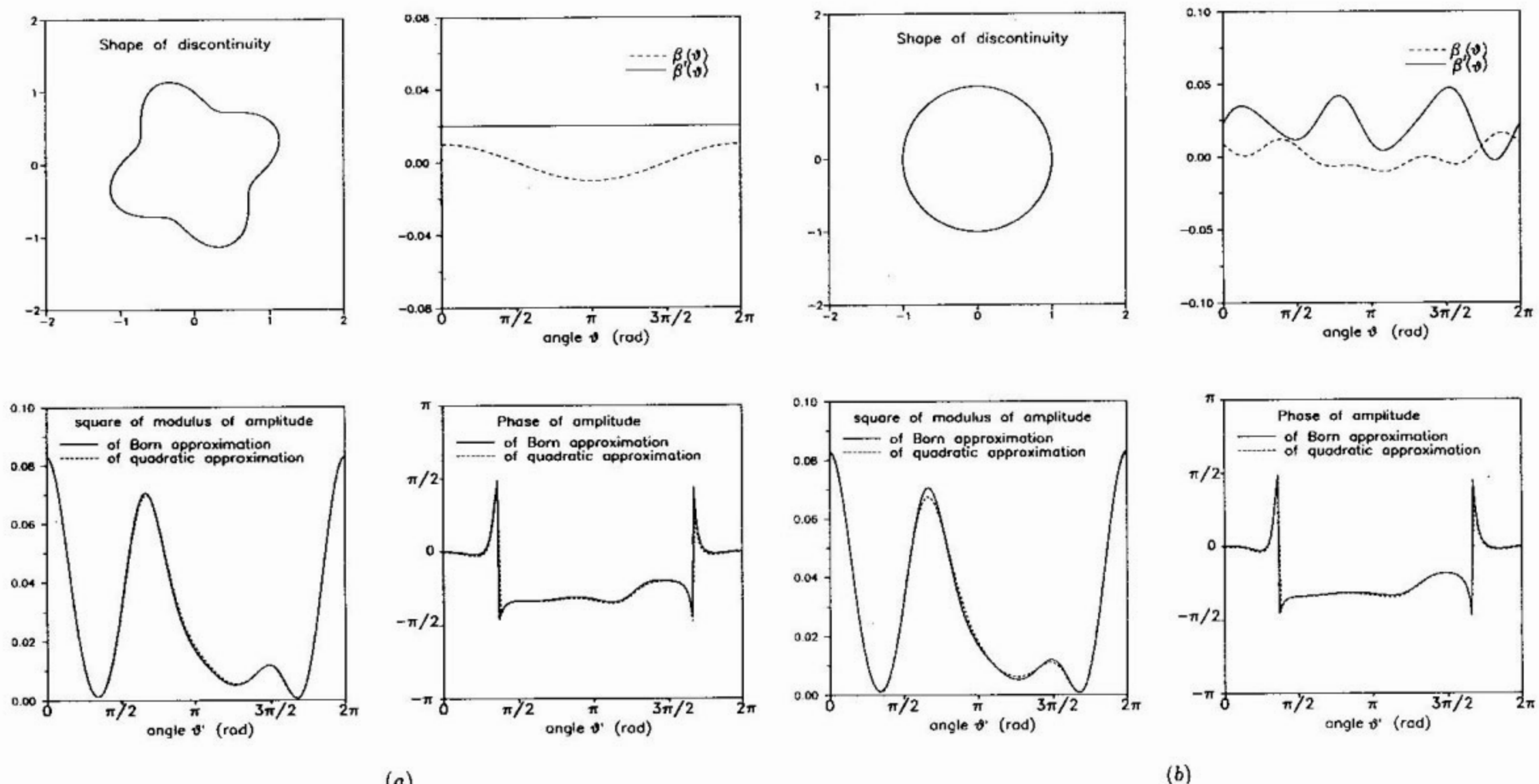
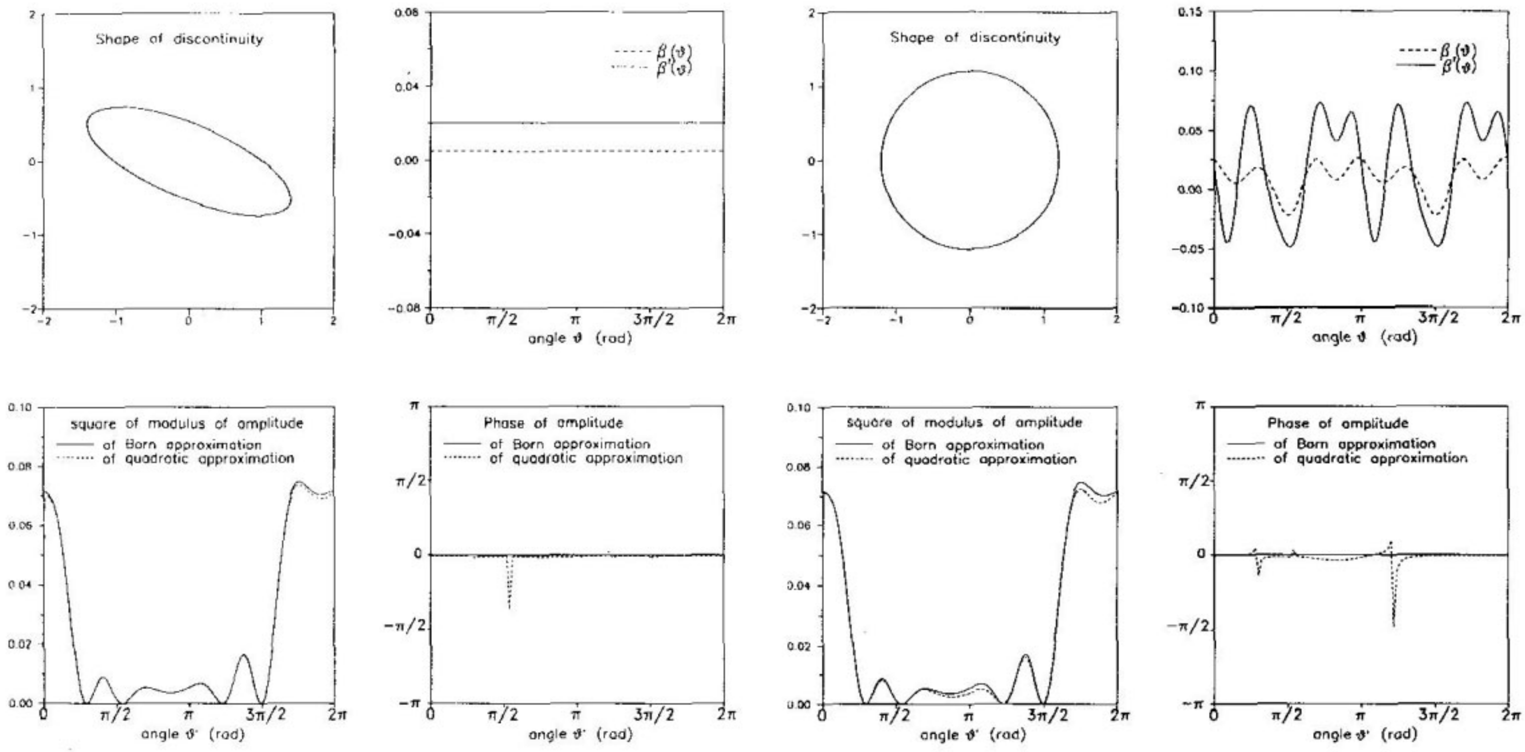


Figure 3. Scattering problems $|k| = 2, 3$. Special data (a) $R = 1 + 0.2$ in 4θ; $\beta = 0.01 \cos\theta$; $\beta' = 0.02$. Radius in (b): $\tilde{R} = 1.0$.



(*a*) (*b*)

Figure 4. Scattering problems $|k|=3$. Special data (*a*): $R=0.75/[0.25\cos^2(\theta+\pi/8)+2.25\sin^2(\theta+\pi/8)]$ $\beta=0.005$; $\beta'=0.02$. Radius in (*b*): $\tilde{R}=1.2$.

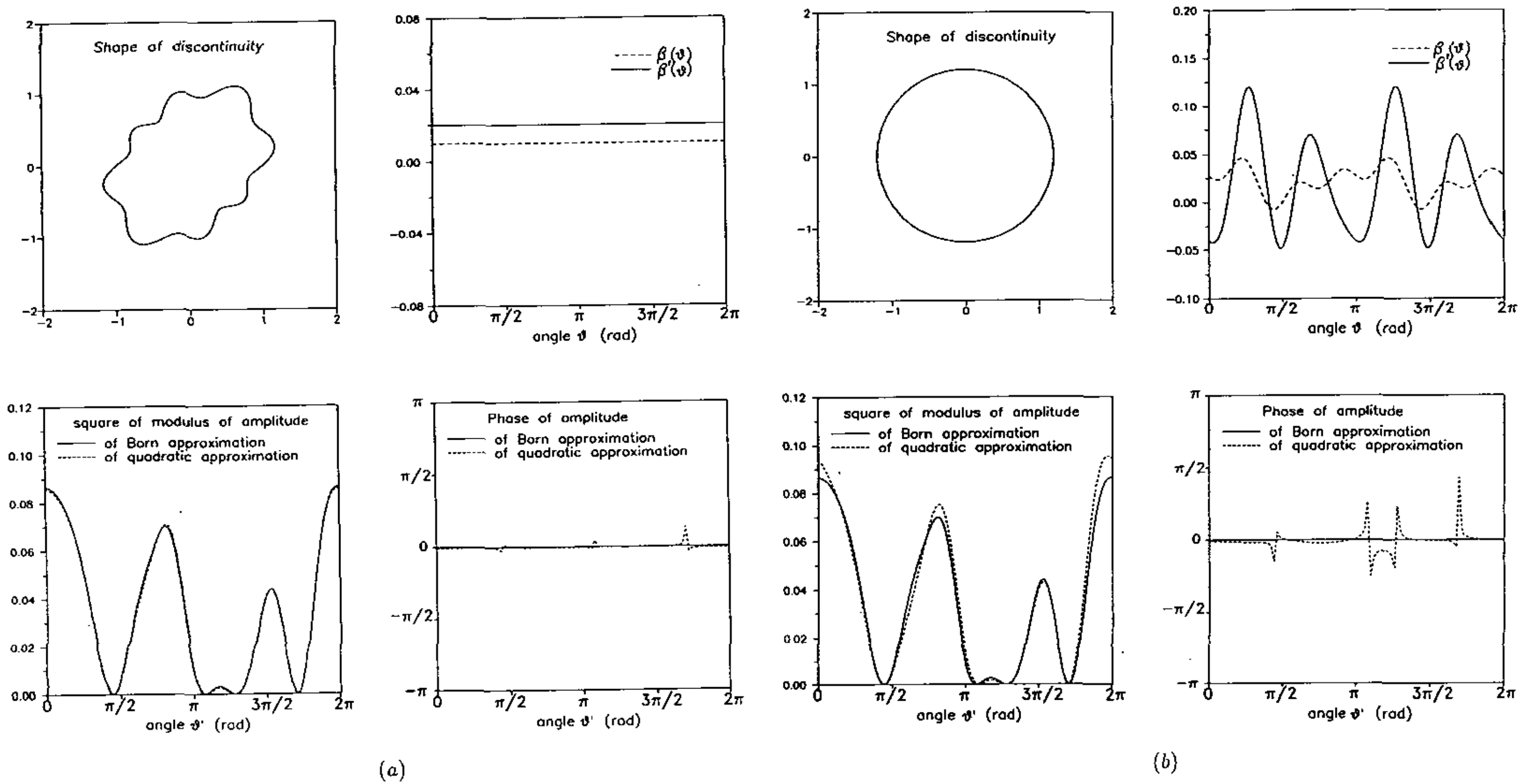


Figure 5. Scattering problems, $|k| = 3$. Special data (a): $R = (1 + 0.2 \sin 2\theta)(1 + 0.1 \sin 8\theta)$; $\beta = 0.01$; $\beta' = 0.02$. Radius in (b): $\tilde{R} = 1.2$.

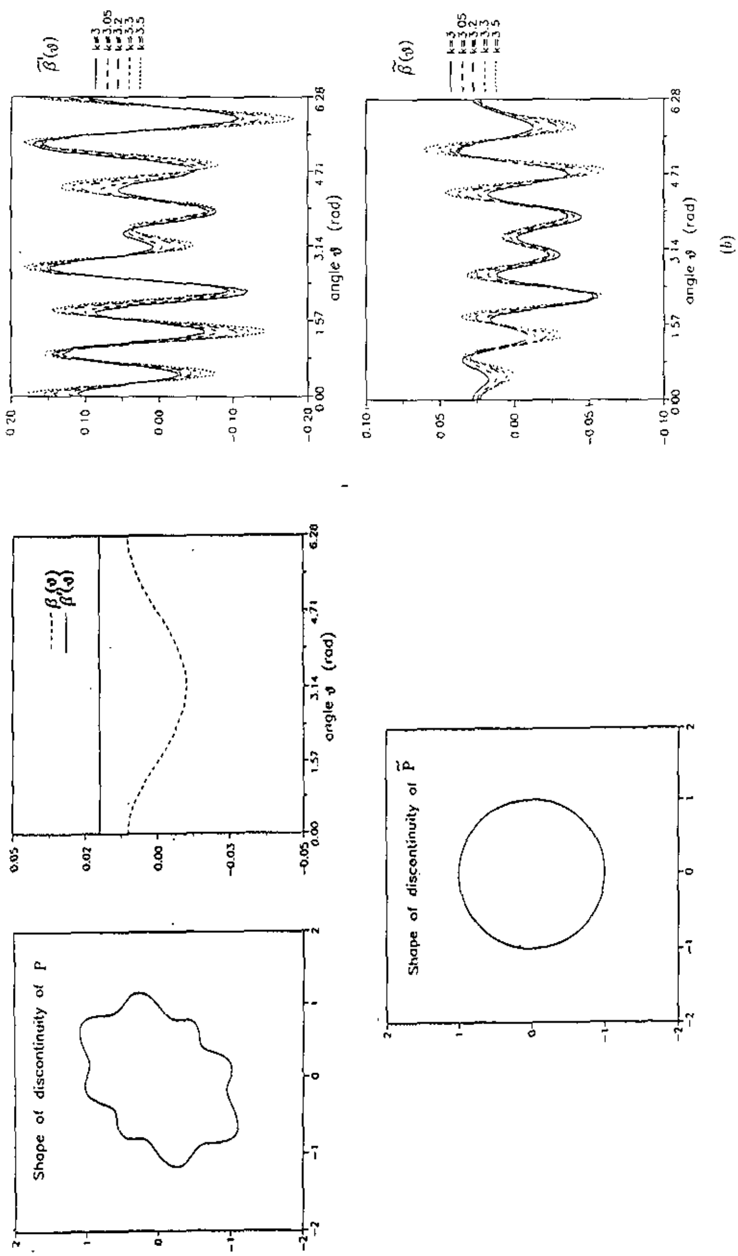


Figure 6. Scattering problem, $k = 3.0$–3.5. Special data (a): $R = (1+0.2 \sin 2\theta)(1+0.1 \sin 8\theta)$. Radius in ($b$): $\tilde{R} = 1$.

To determine $\tilde{\beta}_n$ and $\tilde{\beta}_n$, $n=0,\ldots,+\infty$, the following infinite system must be resolved:

$$A_0=\tilde{A}_0 \qquad B_n=\tilde{B}_n \qquad C_n=\tilde{C}_n \qquad n=1,\ldots,+\infty \tag{3.8}$$

that we truncate at M approximately equal to twice the argument of the Bessel functions. This choice is justified by the fact that small variations around this value do not really modify the values of the scattering amplitude. Achieving the separation between real and imaginary parts, we can reduce the truncated system (3.8) to four blocks, and four matrices are inversed for calculating the Fourier coefficients of $\tilde{\beta}$ and $\tilde{\beta}'$.

We check the equivalence by inserting the computed values of $\tilde{\beta}$ and $\tilde{\beta}'$ into (3.2) and calculating the result by numerical integrations. Notice that this way of proceeding proves the equivalence, no matter how $\tilde{\beta}$ and $\tilde{\beta}'$ were calculated.

3.1. Computer results

Different initial curves, for several values of k, were investigated. Sensitivity with respect to small modifications of k' or $\hat{k}'$ was also checked.

The curves represent a set of equivalent scattering problems for one incident angle at fixed energy at any observation angle θ'. Figures 1–5 illustrate five different pairs of scattering problems. They are characterized by:

(1) The value of $|k|$, i.e. $|k1|$ where 1 is the unit length in the graph that shows the discontinuity curve.

(2) An 'initial' discontinuity curve, of shape chosen, with chosen boundary condition, displayed in the figures labelled (a), where in addition the modulus and phase of the scattering amplitude are represented (calculated by numerical integration).

(3) An equivalent discontinuity curve, of circular shape, with calculated boundary conditions, displayed in the figures labelled (b), where in addition the modulus and phase of the scattering amplitude are represented (calculated by numerical integration).

Figures 6 shows, for a given initial problem, that the calculated values of equivalent $\tilde{\beta}$ and $\tilde{\beta}'$ depend only smoothly on the energy $|k|$ (five values between $|k|=3$ and $|k|=3.5$ are displayed). The calculations were made with Born approximation only but we know from the numerical experiments that the approximation is sound in this case. Note, incidentally, that the existence of an ambiguity does not require that the Born term is a 'good' approximation of the scattering amplitude. When the Born terms of two targets are equal, and the Born series converge *fast enough* (which admittedly is not proved here), there are functional ways to infer from the ambiguity of Born terms that of full amplitudes [4].

4. Physics

The impedance equation (2.1) is a model equation often used to give a first description of several scattering experiments, including elastic waves or electromagnetic waves (and usually in the simplest case of a few homogeneous domains). However, the physical case which exactly fits the equation is that of acoustic waves in a medium characterized by density ρ and Lamé parameter λ, that are both twice differentiable functions of $x\in\mathbb{R}^2$, except on discontinuity surfaces. In this physical problem, the equation for the pressure P at the fixed frequency ω, is

$$\lambda \operatorname{div} \rho^{-1} \operatorname{grad} P+\omega^2 P=0 \qquad P, \rho^{-1}\frac{\partial P}{\partial \nu} \text{ continuous} \tag{4.1}$$

and, in all physical cases, ρ and λ reduce to numbers ρ_0, λ_0 in the most external domain Ω_{N+1}. The problems (2.1) and (4.1) reduce to each other if we set

$$\alpha^2=\rho^{-1} \qquad k^2=\omega^2\rho_0\lambda_0^{-1} \qquad V=\omega^2(\rho_0\lambda_0^{-1}-\rho\lambda^{-1}). \tag{4.2}$$

Let us show now physical scattering problems, and couples among them, which correspond to the cases we have studied. Any one of these scattering problems is characterized by an internal domain Ω_0, where λ and ρ are twice differentiable functions, bounded by the surface S from the external domain Ω_1 where λ and ρ are numbers, λ_0 and ρ_0 (which are assumed equal for all problems). Given β and β' we can determine α and $\partial\alpha/\partial\nu$, or ρ and $\partial\rho/\partial\nu$, on S^-, i.e. their limit values as $x\to S$ from Ω_0. In the formulae we used, we assumed (for the sake of simplicity) that diffuse scattering is missing. Hence λ and ρ should satisfy the constraints in Ω_0

$$\lim \rho, \lim\frac{\partial\rho}{\partial\nu} \text{ given for } x\in S^- \tag{4.3a}$$

$$\lambda, \rho>0 \tag{4.3b}$$

$$V+\alpha^{-1}\Delta\alpha=0 \text{ for } x\in\Omega_0 \tag{4.3c}$$

where V and α are given by (4.2).

Conversely, as soon as (4.3*a*) and (4.3*b*) are satisfied, with β and β' determining (4.3*a*), the scattering problem is completely defined because the Schrödinger chain is defined (standard equivalence). An infinity of continuations of ρ and λ satisfy (4.3). We can see, for instance, simple ways to obtain two, demonstrated here on the simple example of a circular domain $\Omega_0=D(0, R)$.

In the first one, we assume that the Fourier coefficients of ρ are of the form r^2 $\rho_N(r)$, with $\rho_N(r)$ linear as a function of r. We easily construct them from the values at $r=R$. $\lambda(r,\theta)$ is readily extracted from (4.3*c*). The interest of such an interpolation (and other polynomial ones) is that it is not difficult to check (4.3*b*) and also to check that V and $\alpha^{-1}\Delta\alpha$ are small if β and β' are small enough. Although this last property is not required in our case, it could be used to extend ambiguities to cases where the Born approximation applies but (4.3*c*) is not required.

In the second way, we can first assume that ρ and λ are constants up to $r=r_0$, and determine Fourier coefficients such that their variations fit (4.3) in the ring $r_0\leq r\leq R$. Although the consistency limitations (e.g. positivity) will lead us to reject too-thin rings, those acceptable in general can be thin enough to show that ambiguities may be generated by modifying only the scatterer's external part. We can summarize the standard equivalence by saying that 'a scattering experiment at fixed frequency cannot disentangle locally λ and ρ'. Of course, this is a well known result [2], but usually noticed for smooth media only.

Assume now that two scatterers have been constructed in the way described above, and that they satisfy conditions (4.3) for different domains and different reflection and slope coefficients. If the couples $S, \tilde{S}, \beta, \tilde{\beta}, \beta', \tilde{\beta}'$, correspond to one of the examples demonstrated in section 3, and the illumination direction is the right one, the scattering figures are identical. The linearity of our approximation shows that it is even possible to construct an invisible scatterer (for one frequency and one direction of illumination). Fortunately, the numerical experiments show that modifying this direction suppresses the effect, so that it cannot be a moving object.

Our choice of numerical experiments shows that the size of the scatterer and of its main features is of the order of one to a few wavelengths—the ambiguity is partly diffractive. Hence, it is certainly not similar to the well known ambiguities related to shadow boundaries [3], for which λ is much smaller than the scatterer size. Neither is it similar to the fixed energy ambiguities in quantum mechanics, which do not hold for a finite scatterer (although they may be a source of instabilities in its reconstruction) [4] and hold even for an infinity of illumination directions. On the other hand, our numerical experiments are in a domain such that 'Heisenberg uncertainties' can also be discarded as being mainly responsible for ambiguities. Needless to say, one may always think and say that ambiguities are produced by a mixture of shadow boundaries, diffractive uncertainties, standard equivalence, these being extreme clear cut cases, but the mixture we have shown here has its own interest: it shows that in diffractive tomography, as in ray tomography, one frequency and one illumination direction are not enough to determine a medium by acoustic scattering, *even if all observation directions are used.* Furthermore, the fact that our ambiguity is continuous suggests that there are special cases (we did not demonstrate them) where two (or maybe a few) illumination directions at a single frequency are not sufficient to determine a medium. Our examples are of course consistent with known theorems and should be viewed in the context of mathematic analysis of boundary measurements [5], and of tomograpnic experiments [6] as a warning in certain non-destructive sensings.

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